Self-Similar Log-Periodic Structures in Western Stock Markets from 2000[*]

M. Bartolozzi,[1,†] S. Drożdż,[2,3] D. B. Leinweber,[1] J. Speth,[4] and A. W. Thomas[5,1]

[1]*Special Research Centre for the Subatomic Structure of Matter (CSSM),*
University of Adelaide, Adelaide, SA 5005, Australia

[2]*Institute of Nuclear Physics, Radzikowskiego 152, 31–342 Kraków, Poland*

[3]*Institute of Physics, University of Rzeszów, PL-35-310 Rzeszów, Poland*

[4]*Institute für Kernphysik, Forschungszentrum Jülich, D–52425 Jülich, Germany*

[5]*Jefferson Laboratory, 12000 Jefferson Ave., Newport News, VA 23606, USA*

(Dated: February 2, 2008)

Abstract

The presence of log-periodic structures before and after stock market crashes is considered to be an imprint of an intrinsic discrete scale invariance (DSI) in this complex system. The fractal framework of the theory leaves open the possibility of observing self-similar log-periodic structures at different time scales. In the present work we analyze the daily closures of four of the most important indices worldwide since 2000: the DAX for Germany and the Nasdaq100, the S&P500 and the Dow Jones for the United States. The qualitative behaviour of these different markets is similar during the temporal frame studied. Evidence is found for decelerating log-periodic oscillations of duration about two years and starting in September 2000. Moreover, a nested substructure starting in May 2002 is revealed, bringing more evidence to support the hypothesis of self-similar, log-periodic behavior. Ongoing log-periodic oscillations are also revealed. A Lomb analysis over the aforementioned periods indicates a preferential scaling factor $\lambda \sim 2$. Higher order harmonics are also present. The spectral pattern of the data has been found to be similar to that of a Weierstrass-type function, used as a prototype of a log-periodic fractal function.

---

[*]Colour Manuscript: http://www.physics.adelaide.edu.au/cssm/econophysics/lp_structures_col.ps

[†]Electronic address: mbartolo@physics.adelaide.edu.au

I.   INTRODUCTION

It is well known that many physical systems undergo phase transitions around specific critical points in the parameter space [1, 2]. Near these points the system is strongly correlated and many characteristic quantities can well be approximated by power laws, related to the scale-invariance of the system in that state. If we assume that $\phi(t)$ is an observable near a critical point, t_c, as for example the susceptibility of the Ising model near the critical temperature, for a change of scale $t \rightarrow \lambda t$ we have

$$\phi(\lambda t) = \mu \phi(t), \tag{1}$$

where $\mu = \lambda^\alpha$ since $\phi(t) \sim t^\alpha$. The power law is a solution of Eq.(1) for $\forall \lambda$.

A weaker version of the over mentioned scale invariance is the *discrete-scale invariance* (DSI) [3]. In this case the system becomes self-similar only for an infinite but countable set of values of the parameter λ. That is Eq.(1) holds only for $\lambda = \lambda_1, \lambda_2...$ where in general $\lambda_n = \lambda^n$. In this case λ represents a preferential scaling factor that characterizes a hierarchical structure in the system. The solution of Eq.(1) can be written in a more generic form that accounts also for a possible discrete scale invariance:

$$\phi(t) = t^\alpha \; \Theta \left(\frac{\ln(t)}{\ln(\lambda)} \right), \tag{2}$$

where Θ is an arbitrary periodic function of period 1. Using a first order Fourier expansion on Eq.(2) and writing $t \rightarrow |t_c - t|$ we obtain

$$\phi(t) = A + B|t_c - t|^\alpha + C|t_c - t|^\alpha \cos(\omega \ln |t_c - t| - \varphi), \tag{3}$$

where $\omega = 2\pi / \ln(\lambda)$. The dominant power law behaviour, a hallmark of all critical phenomena, and the log-periodic corrections to the leading term are the main features of Eq. (3).

Sornette, Johansen and Bouchaud [4, 5] first pointed out how different price indices in the stock market show a power law increase with superimposed accelerating oscillations just before a crash. The remarkable fact was that the log-periodic formula (3), derived for DSI systems, provided a very good approximation for this empirical fact. This led them to conjecture about the existence of a critical point in time, t_c, for which the market can undergo a phase transition (crash).

According to this framework the stock market is seen as a self-organized system that drives itself toward a critical point. Just as the Ising model has a parameter governing the

2

temperature of the system and a critical temperature where the system undergoes a phase transition, it is postulated that the market also has such an underlying parameter which takes the critical value at time t_c. The appearance of log-periodic oscillations has been related to a *discrete* hierarchal structure of the traders. Near the critical point, when the market is very compact and unstable, every perturbation can spread throughout the system: a common decision to sell by a certain group in the hierarchy of traders can trigger a *herd* effect, leading to a crash. This concept, in a way, is similar the *self-organized criticality* proposed by Bak, Tang and Wiesenfeld [6], studied recently in Ref. [7].

Since the first paper by Sornette, Johansen and Bouchaud [4] many physicists have been attracted by the idea of phase-transitions in a self-organized stock market [8, 9, 10, 11], even if criticisms have been also raised [12, 13, 14]. A recent review on the subject can be found in Ref. [15].

An intriguing scenario has been proposed by Drożdż and coworkers [10, 11]. Inspired by theoretical consistency arguments, they found empirical evidence that short time log-periodic structures can be nested within log-periodic structures on a larger scale. The appearance of these self-similar periods, one inside the other, has been related to the underlying fractal nature of the DSI, giving rise to a multi-scale log-periodicity. Moreover, the existence of a preferential scaling ratio $\lambda \sim 2$ has been pointed out for both the leading pattern and the related sub-structures. This last fact led them to formulate a universality hypothesis for the parameter λ and as a consequence they fixed *a priori* the frequency of the oscillations to correspond to $\lambda = 2$. In this way, the predictive power of Eq. (3) increases considerably [10, 11]. Further evidence of embedded sub-structures has been reported recently by Sornette and Zhou [16, 17].

Log-periodic patterns have been observed not only in bullish periods of the market but also during the "antibubbles", or bearish periods, that follow a market peak [11, 18, 19, 20]. An example of these log-periodic oscillations has been documented during the long period of recession experienced by most of the world stock markets since the middle of 2000 [20]. In this period, which ends approximately (in our understanding) in the first months of 2003, all the most important markets world-wide are remarkably synchronized. A simple plot of the logarithmic indices, $P(t)$, is provided Fig. 1. We believe this is sufficient to convince the reader of this behaviour. It is nothing but an expression of the growing globalization of the modern economy [21].
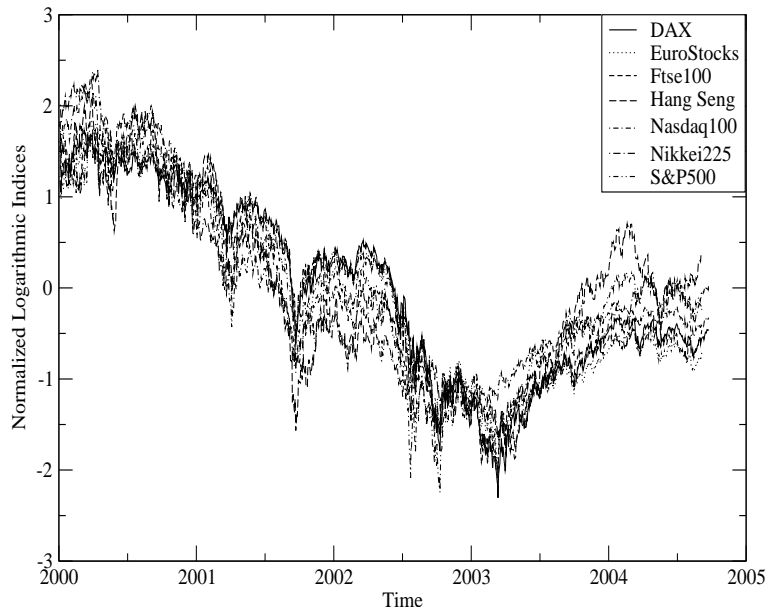
FIG. 1: Time series of the logarithm of the most important indices world-wide since 2000. The time series have been appropriately normalized, $P(t) \rightarrow \frac{P(t)-<P(t)>}{\sigma(P(t))}$, where $< ... >$ denotes the average over the period under consideration and σ is the standard deviation. The synchronization of the indices during the recession period, and further on for some, is an expression of the modern globalized economy. The year tick in the graph marks, from now on, the position of the 1st of January of the year itself.

In the present work we focus on the study of the daily closures of three of the most important international indices from 2000 until the end of September 2004: the DAX for the German market and the S&P500, the Nasdaq100 and the Dow Jones[§] for the American market. We confirm (within our DSI model of Eq. (2)), the existence of a main log-periodic structure starting in September 2000 and ending in November 2002 for all the indices in Sec. II. Moreover, for the first time, we identify a clear bearish sub-structure, starting around the 15th of May 2002 and ending one year later. Ongoing log-periodic oscillations with the same characteristic frequency, starting in January 2004, also are reported.

We also address the question of a possible universality of the power law exponent α,

---

[§] The Dow Jones index has been included for historical, more that economic, reasons. In fact, it constitutes just a small sub-set of the S&P500 basket but, nevertheless, it is considered by many to be a good indicator for the health of the economy.

related to the trend of the time series. We then convert the S&P500, Nasdaq100 and the Dow Jones in Euros while the DAX is expressed in American Dollars. The conversion, while leaving the oscillations unaltered, seriously distorts the trends. Therefore, no universal characteristic can be claimed for this parameter.

A Lomb analysis of the main structure and the relative sub-structure, presented in Sec. III, reveals that the dominating frequency of both is related to a common value of λ that is $\lambda \sim 2$. A second relevant harmonic, at a frequency double that of the fundamental, is also present. These results confirm the fractal hypothesis of Drożdż *at al.* [10, 11]. Further indications pointing to self-similar log-periodicity have been found using a Lomb analysis of a Weierstrass-type function [22, 23], taken as prototype of a log-periodic fractal function. The relevance of such a function for stock market log-periodic criticality was suggested for the first time in Ref. [10].

II.   LOG-PERIODICITY AND SUB-STRUCTURES

The DSI model of Eq. (3) is used to fit the logarithm of the DAX, Nasdaq100 S&P500 and Dow Jones. Considering that the periodic function Θ is arbitrary, we take the modulus of the cosine as it provides a better representation of the data [11] than the cosine itself. The fitting procedure that we use is borrowed from Ref. [4]. According to that we express the parameters from A, B and C of Eq. (3) as a function of α, ω (or λ), φ and t_c by imposing the constraint that the cost function, χ^2, has null derivative with respect to them. Following the method by Drożdż *at al.* [10, 11], we fix the parameter λ, considering it to be a universal constant. In particular we choose $\lambda = 2$. If this assumption turns out to be confirmed, then the predictive power of the model will considerably be increased. Moreover, as we are interested here in bearish periods only and not in predicting the most probable crash time, we introduce a further simplification by adjusting t_c visually. In this way there are only two parameters left to explore: α and φ. The results of the fits are shown in Figs. 2, 3, 4 and 5 for the DAX, the Nasdaq100, the S&P500 and Dow Jones respectively.

A log-periodic structure starting around the 1st of September 2000 and finishing in November 2002 is clearly evident for all the sets of data considered. Moreover a nested sub-structure starting around the 15th of May 2002 is also visible. Log-periodic oscillations also characterize the present state of the market. A possible origin of this behaviour can be
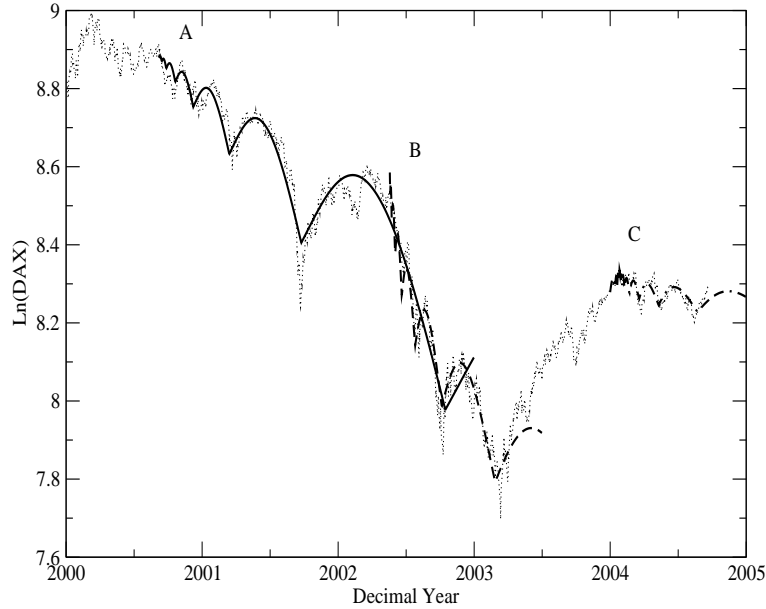
FIG. 2: Time series of the DAX from 1/1/2000 until 22/9/2004. A log-periodic structure of the approximate duration of two years and starting in September 2000 is highlighted by the solid curve labeled (A). In this case we fix t_c =1/9/2000 in Eq. (3). A one year sub-structure is visible starting in May 2002, as illustrated by the dashed curve (B) (t_c =16/5/2002). The dashed dotted curve labeled (C) (t_c =26/1/2004) is related to the ongoing log-periodic oscillations. For all the fits we fixed $\lambda = 2$.

localized at the end of January 2004.

At this point it is important to emphasize that the previous fits are obtained for a preferential scaling coefficient, $\lambda = 2$. This is already a good indication of the universal nature of the hierarchical scaling in stock markets.

Another interesting point regards the universality of the parameter α, related to the main trend of the time series. In order to have some insight into this direction we repeat the previous fits after the conversion of the various indices into different currencies. That is we transform the DAX from Euros to American dollars, and the Nasdaq100, S&P500 and Dow Jones from American dollars to Euros. The results are shown in Figs. 5, 6, 7 and 8.

It appears clear from the plots that, while the oscillatory structures are unaltered by the currency conversion, the trends experience a serious distortion and therefore we cannot extract a universal characteristic for the exponent α. On the other hand, one might wonder to
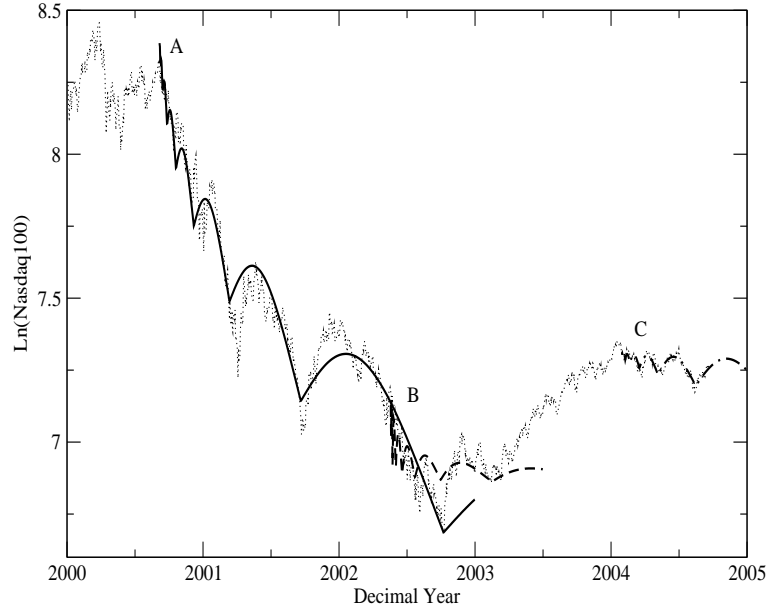
6

FIG. 3: Time series for the Nasdaq100 index from 1/1/2000 until 22/9/2004. The critical times and the coding are the same as in Fig. 2. The sub-structure starting in May 2002 is not as evident as for the S&P500 or DAX.
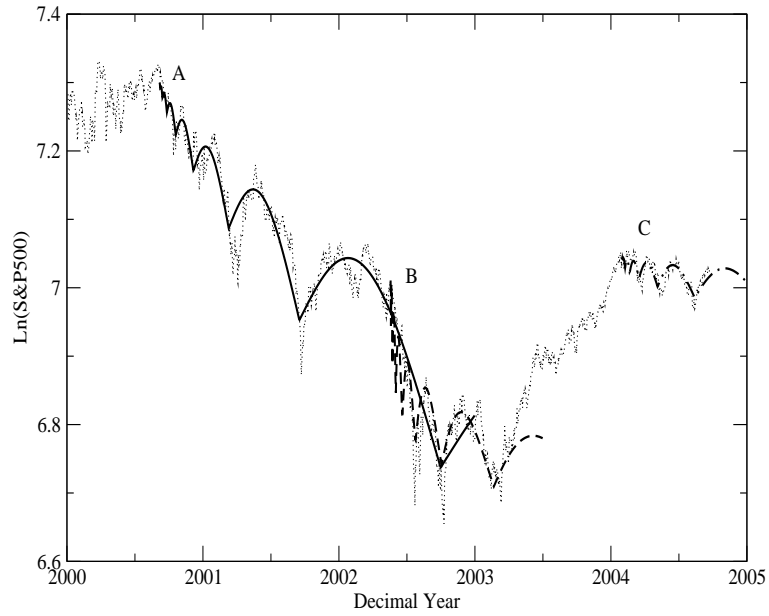

FIG. 4: Time series for the S&P500 index from 1/1/2000 until 22/9/2004. The critical times and coding are the same as in Fig. 2.
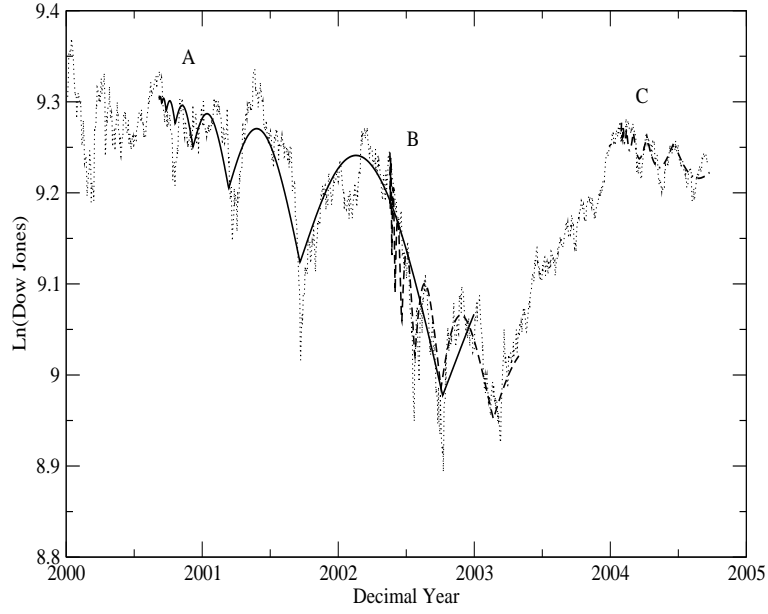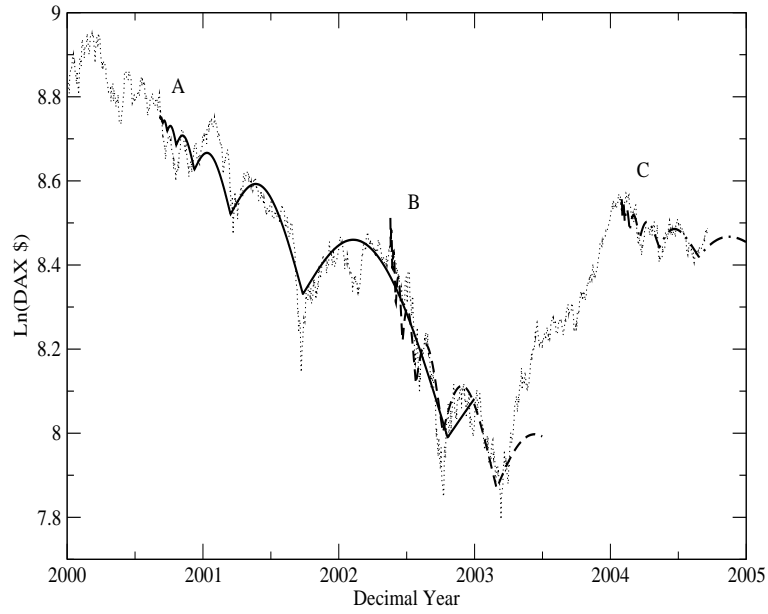
FIG. 5: Time series for the Dow Jones index from 1/1/2000 until 22/9/2004. The critical times and coding are the same as in Fig. 2.

FIG. 6: The DAX expressed in American dollars. Features are as described in Fig. 2.
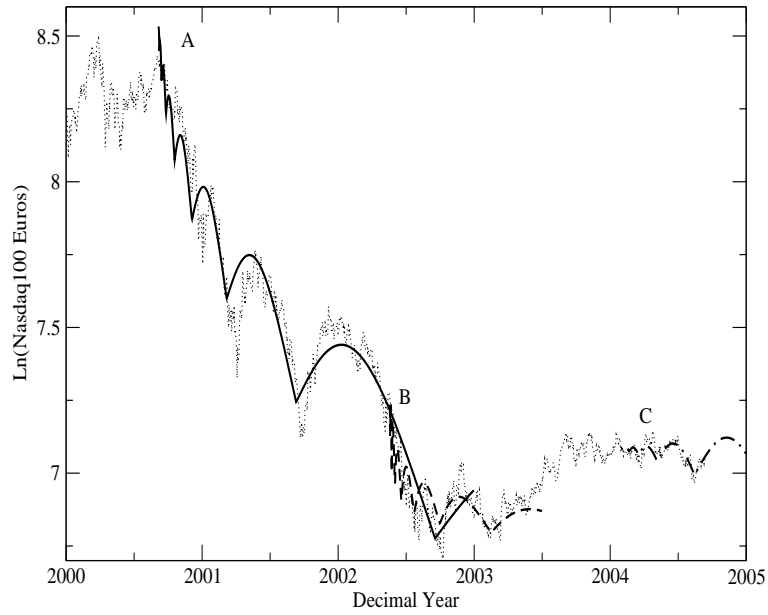
8

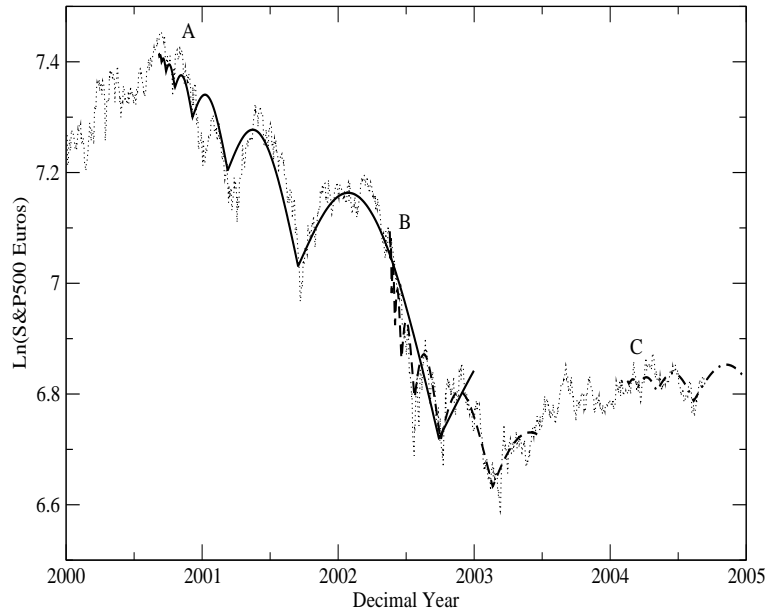FIG. 7: The Nasdaq100 expressed in Euros. Features are as described in Fig. 2.

FIG. 8: The S&P500 converted from American dollars to Euros. Features are as described in Fig. 2.
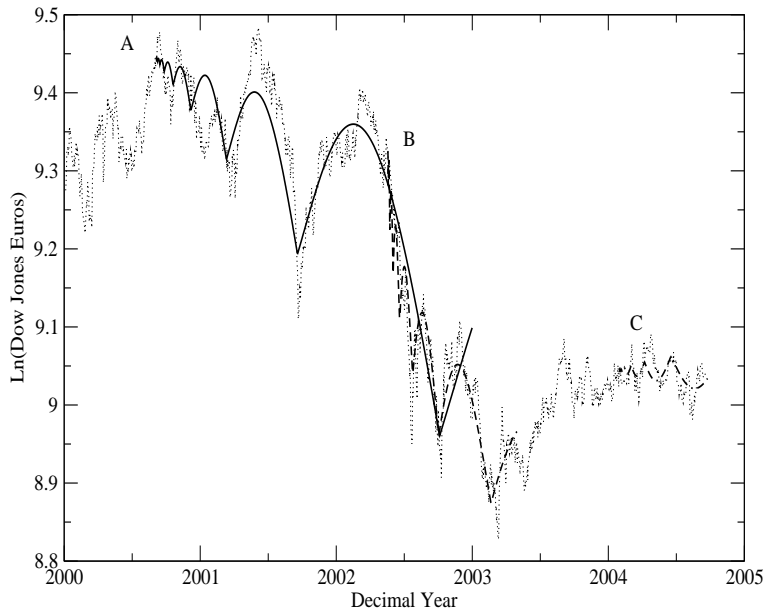
9

FIG. 9: The Dow Jones converted from American dollars to Euros. Features are as described in Fig. 2.

what extent the market dynamics are digested in a currency other than the native currency[‡].

III.   A NON-PARAMETRIC APPROACH: THE LOMB ANALYSIS

In order to justify our assumption of $\lambda = 2$ we perform a non-parametric test on the angular frequency value of the log-periodic oscillations. Following the method proposed by Johansen *et al.* in Ref. [25], we analyze the time series of residuals, $R(t)$, obtained by removing the leading power law trend in the logarithm of the price, $P(t)$, according to

$$R(t) = \frac{P(t) - A - B|t_c - t|^\alpha}{C|t_c - t|^\alpha}. \tag{4}$$

If the model of Eq. (3) reproduces the behaviour of the market correctly then the residual dependence on in the variable $\ln(|t_c - t|)$ must be a cosine function and a spectral analysis should reveal a high peak corresponding to the angular frequency ω.

[‡] The problem of a change in the currency of the S&P500 index has been addressed also in Ref. [24]. In this case it was argued that the main source of distortion of this index is related to the depreciation of the American dollar due to the feedback action of the Federal Reserve Bank.
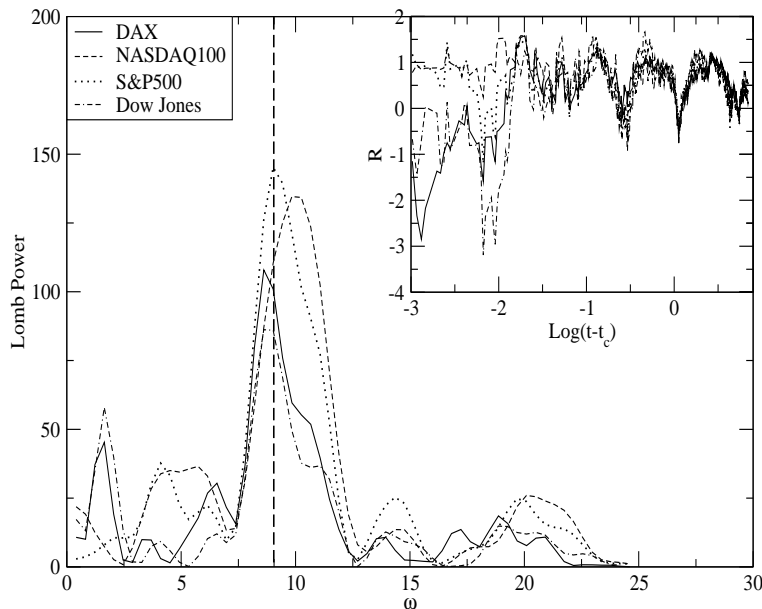
10

FIG. 10: Lomb analysis for the DAX (solid line), Nasdaq100 (dashed line), S&P500 (dotted line) and Dow Jones (dashed-dotted line) during the period from 1/9/2000 to 30/12/2002. A main frequency around $\tilde{\omega} = 9.06$ ($\lambda = 2$), dashed vertical line, is clearly evident. Another important harmonic contribution can be seen at $\omega \approx 2\tilde{\omega}$. In the insert the residuals, with the same coding, are plotted.

Once we have obtained the residuals, shown in the inserts of Figs. 8, 9 and 10, we apply a spectral decomposition of these signals according to the Lomb algorithm [26]. This spectral algorithm makes use of a series of local fits using a cosine function with a phase and provides some practical advantages, compared to the classical Fourier transform, when the data under examination are not evenly sampled, as in our case. The results of the Lomb analysis for the periods under consideration are presented in Figs. 8, 9 and 10. In these plots the angular frequency corresponding to $\lambda = 2$, that is $\omega \approx 9.06$, is represented by a vertical dashed line. The results of the analysis show, for all the periods, a dominating peak in the vicinity of $\lambda = 2$. This fact brings more evidence to the existence of a universal scaling factor $\lambda \approx 2$. It is also important to underline how the Lomb analysis shows, in most of the cases, a second main frequency that is about two times the leading frequency.

We can also assign a confidence level to the main peaks found in the analysis. Following the technique proposed in Ref. [27], the ratio between the two highest peaks in the Lomb
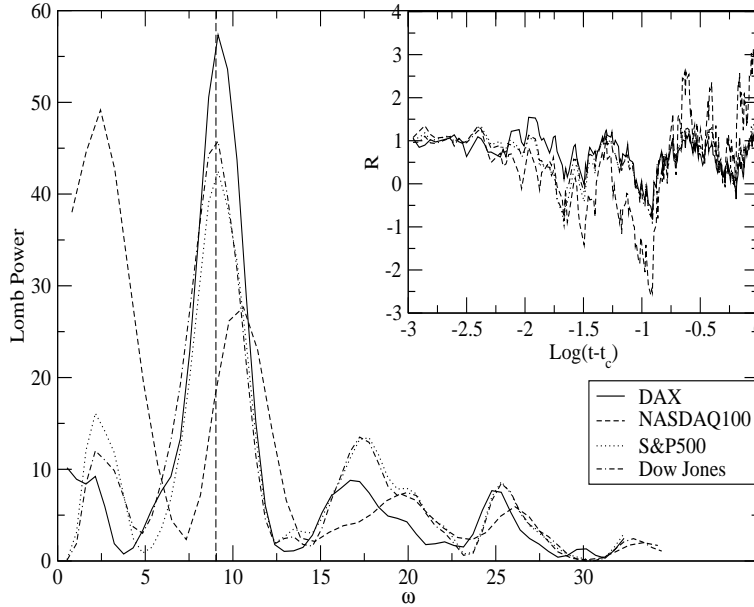
11

FIG. 11: Lomb analysis for the DAX (solid line), Nasdaq100 (dashed line), S&P500 (dotted line) and Dow Jones (dashed-dotted line) during the period from 16/5/2002 to 3/5/2003. Regular high order harmonics are still present. As already seen from the fit, the log-periodic behaviour of the Nasdaq100 index is not as clear as for the other indices. The residuals are presented in the insert.

periodogram is used to give an estimation of the significance level of the higher peak. In selecting the peaks for the ratio are excluded the higher-order harmonics, multiples of the fundamental. All the ratios found from our analysis are clustered in a range approximately between 3 and 6, except for the Nasdaq100 index of Fig. 9. Even for a ratio of about 3, the confidence level is higher than 99% assuming Gaussian noise [27]. If, instead, we assume that the noise is temporally correlated, then a fractional Brownian motion [28] with Hurst exponent, H, at the worst, unrealistic, case of $H = 0.9$, provides a confidence level which remains greater than 80% [27]. Hence all the peaks found by the Lomb analysis show a high statistical significance, with the only exception being that of the Nasdaq100 where the first peak has a low confidence level.

In order to have a better understanding of the spectral patterns just found, we test the same method of analysis on a Weierstrass-type function [22, 23]. The Weierstrass-type functions are a particular solution of the discrete renormalization group equation for critical phenomena [2, 23]. Defined in the interval [0,1], these functions are characterized by a self-
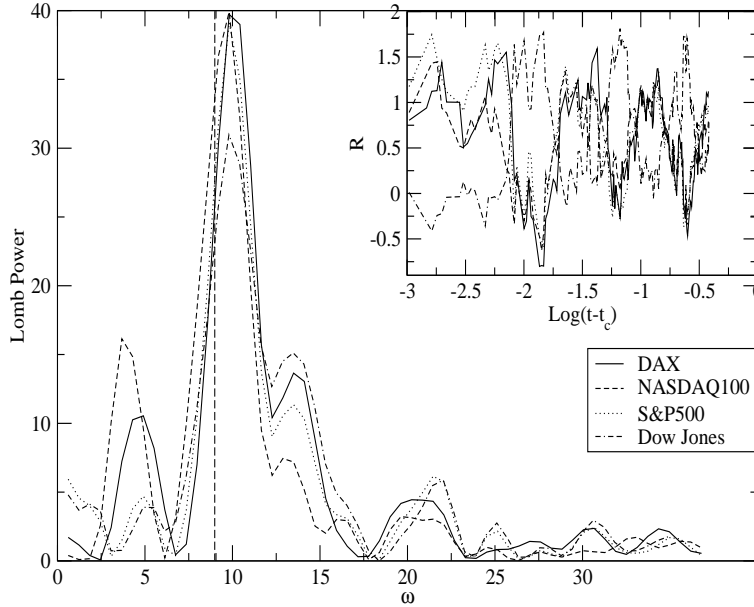
12

FIG. 12: Lomb analysis for the DAX (solid line), Nasdaq100 (dashed line), S&P500 (dotted line) and Dow Jones (dashed-dotted line) during the period from 26/1/2004 to 22/9/2004. A main frequency along with other higher order harmonics are revealed. In this case the phase of the residuals for the Dow Jones differs from the other indices, reflecting a less then perfect synchronization.

similar hierarchy of log-periodic structures accumulating at a critical point t_c (t_c can be 0 or 1 according to particular choices of the parameters). Zhou and Sornette have shown that Weierstrass-type functions can provide a good approximation for the bearish period of the stock market starting from 2000 [29].

The following Weierstrass-type function [2, 23] has been used as a test for the Lomb method:

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{(2-D)n}} \exp[-\lambda^n t \cos(\gamma)] \cos[\lambda^n t \sin(\gamma)], \tag{5}$$

where $\gamma \in [0, \pi/2]$ is a parameter that fixes the oscillatory structures. If $\gamma = \pi/2$ then the parameter D corresponds to the fractal dimension of the function. For $\gamma < \pi/2$ the function becomes smooth and it is no longer fractal [2, 23] but preserves the large scale log-periodic oscillation. Characteristic curves are illustrated in Fig. 11.

Once we have chosen the test function, the same procedures as used for the stock market time series are applied to the artificial time series generated by Eq. (5) with $\lambda = 2$. fixed.
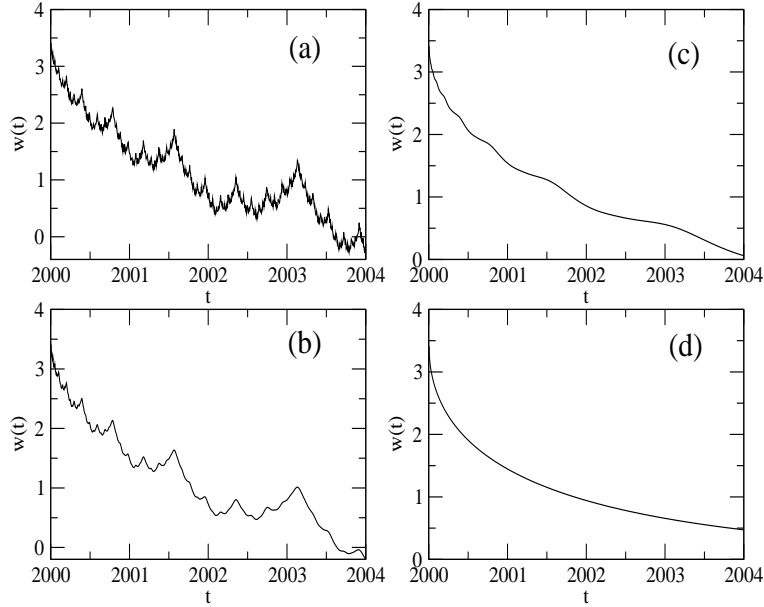
13

FIG. 13: The Weierstrass-type function of Eq. (5) for (a) $\gamma = \pi/2$, (b) $\gamma = 0.93\pi/2$, (c) $\gamma = 0.90\pi/2$ and (d) $\gamma = 0$. For all the plots $D = 1.5$ and $\lambda = 2$. The sum in Eq. (5) has been truncated at $N = 32$ because the function does not change significantly beyond this value of N. The time axis has been rescaled, as the Weierstrass-type function is defined only for $t \in [0,1]$. In the present plots the function is fractal only in (a).

In this case the Lomb analysis reveals for the fractal case ($\gamma = \pi/2$), apart from the clear peak at the main frequency, other smaller, high frequency harmonics regularly spaced, as illustrated in Fig. 12.

The higher frequency periodic peaks are a manifestation of the fractality of the function itself. In fact, for $\gamma < 1$ those frequencies are absent. The high frequency harmonics, in the pure fractal case ($\gamma = \pi/2$), are a reflection of the self-similarity of the function at different scales.

The real data has similar high frequency modes that have about the same spacing as the ones artificially obtained with the Weierstrass-type function. We can also conjecture that these modes are related, as in the previous case, to self-similar structures in the time series. Of course, because of the lack of points and noise effects, these harmonics are not as clear as for the Weierstrass-type function.

The similarity in the spectral pattern between real and artificial data is an indication
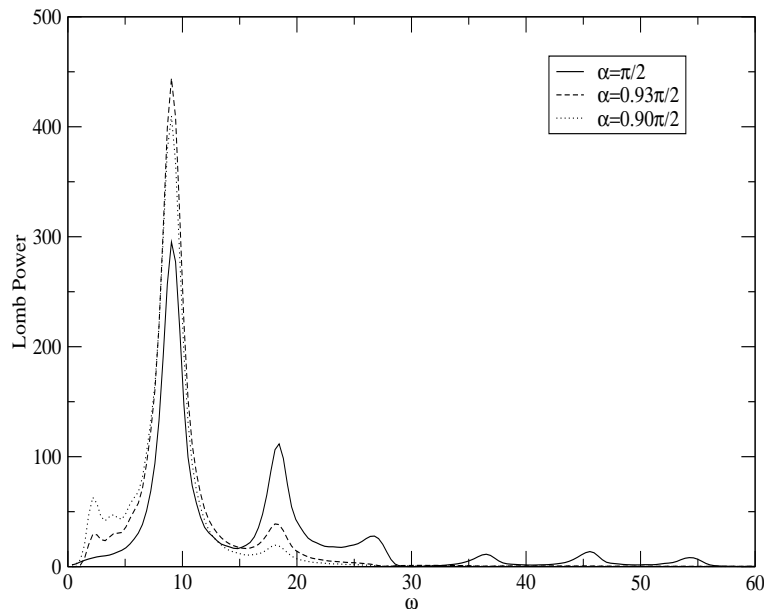
14

FIG. 14: Lomb analysis for the Weierstrass-type function of Eq.(5). The self-similarity of the function in the fractal case ($\gamma = \pi/2$) is reflected in the regularity of the high order harmonics. Once we smooth the function, the high order harmonics related to the fractality disappear while the dominant frequency of the log-periodic leading term is unaltered.

of the existence of self-similar structures at different scales in stock market time series, providing further support for the fractal framework of Drożdż and coworkers [10, 11]. The sub-structure starting in May 2002 is a clear example of self-similarity in stock market dynamics.

IV. CONCLUSIONS

In the present work we have shown that, at least, three clear log-periodic periods have characterized, and still characterize, the behaviour of some of the most important indices worldwide since the year 2000. Moreover, one of the log-periodic structures found is embedded in a longer one, interestingly, both in the decelerating market phase. This finding supports the hypothesis of self-similar log-periodicity proposed by Drożdż and coworkers [10, 11]. A non parametric analysis over these periods has also been performed. The results of the analysis confirm the existence of log-periodic structures. Moreover, we found further evi-

dence for a preferential scaling factor of $\lambda \sim 2$. The presence of a higher order harmonic at a frequency that is double the fundamental can also be related to the fractal structure of the time series. A test on a Weierstrass-type function supports this hypothesis. We have also investigated a possible universality of the power law index α. For this purpose we have converted the price time series to different currencies, namely the DAX from Euros to American Dollars and the Nasdaq100, the S&P500 and the Dow Jones from American dollars to Euros. While the log-periodic oscillations remain unaltered by this procedure, the trends come to be seriously distorted and no universality can be claimed.

V.   ACKNOWLEDGEMENT

[1] H.E. Stanley, *Introduction to Phase Transitions and Critical Phenomena*, (Oxford University Press, New York, 1971).

[2] D. Sornette, *Critical Phenomena in Natural Sciences*, (Springer-Verlag, Berlin, 2004).

[3] D. Sornette, Phys. Rep. **297**, 239 (1998).

[4] D. Sornette, A. Johansen, J.-P. Bouchaud, J. Phys. I France **6**, 167 (1996).

[5] D. Sornette, A. Johansen, Physica A **245**, 411 (1997).

[6] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[7] M. Bartolozzi, D.B. Leinweber, A.W. Thomas, Physica A, in press, preprint cond-mat/0405257.

[8] J.A. Feigenbaum, P.G.O. Freund, Int. J. Mod. Phys. B **10**, 3737 (1996).

[9] N. Vandevalle, M. Ausloos, Ph. Boveroux, A. Minguet, Eur. Phys. J. B **4**, 139 (1998); N. Vandevalle, Ph. Boveroux, A. Minguet, M. Ausloos, Physica A **255**, 201 (1998).

[10] S. Drożdż, F. Ruf, J. Speth, M. Wójcik, Eur. Phys. J. B **10**, 589 (1999).

[11] S. Drożdż, F. Grümmer, F. Ruf, J. Speth Physica A **324**, 174 (2003).

[12] L. Laloux, M. Potters, R. Cont, J.P. Aguilar, J.P. Bouchaud, Europhys. Lett. **45**, 1 (1999).

[13] K. Ilinski, Int. J. Mod. Phys. C **10**, 741 (1999).

[14] J.A. Feigenbaum, Quantitative Finance **1**, 346 (2001).

[15] D. Sornette, *Why Stock Market Crash*, (Princenton University Press, Princenton, 2003).

[16] D. Sornette and W.-X. Zhou, Quant. Finance **3**, C39 (2003).

[17] W.-X. Zhou. and D. Sornette, Physica A **337**, 586 (2004).

[18] A. Johansen, D. Sornette, Eur. Phys. J. B **17**, 319 (2000).

[19] D. Sornette and W.-X. Zhou, Quantitative Finance **2**, 468 (2002).

[20] W.-X. Zhou. and D. Sornette, Physica A **330**, 543 (2003).

[21] S. Drożdż, F. Grümmer, F. Ruf, J. Speth, Physica A **294**, 226(2001).

[22] M.V. Berry and Z.V. Lewis, Proc. R. soc. lond. A **370**, 459 (1980).

[23] S. Gluzman and D. Sornette, Phys. Rev. E **65**, 036142 (2002).

[24] W.-X. Zhou. and D. Sornette, Physica A **348**, 428 (2005); http://www.ess.ucla.edu/faculty/sornette/prediction/index.asp#prediction.

[25] A. Johansen, D. Sornette and O. Ledoid, Journal of Risk **1 (4)**, 5-32 (1999).

[26] W.H. Press, S.A. Teulosky, W.T. Vetterlong and B.P. Flannery, *Numerical Recipies in Fortran*, (Cambridge University Press, Cambridge, 1994).

[27] W.-X. Zhou. and D. Sornette, Int. J. Mod. Phys. C **13**, 137 (2002).

[28] J. Feder, *Fractals*, (Plenum Press, New York & London, 1988).

[29] W.-X. Zhou. and D. Sornette, Physica A **330**, 584 (2003).